ASSIGNMENT PROBLEMS

Introduction :→ In practical field we are sometimes faced with a type of problem which consists in assigning men to offices, jobs to machines, classes in a school to rooms, drivers to trucks, delivery trucks to different routes or problems to different research teams. etc. in which the assignees possess varying degree of efficiency, called cost or effectiveness. The basic assumption of this type of problem is that one person can perform ~~the total cost or effectiveness or maximizes it~~ one job at a time. An assignment plan is optimal if it minimizes the total cost or effectiveness or maximizes the profit of performing all the jobs. For example, the manager of a firm may be interested in finding the best assignment of m jobs to him m employees. Thus the assignment problem is a special type of transportation problem in which the objective is to optimize the effect of allocating a number of jobs to an equal number facilities; the jobs and facilities in assignment problem represent origins and destinations in transportation problem. As only one job is assigned to one facility, the cost matrix is always a square.

If c_{ij} be the cost of assigning the i-th job to the j-th facility, we can represent the cost matrix in the following table:

Facilities

Jobs	1	2	3	...	m	a_i
1	c_{11}	c_{12}	c_{13}	...	c_{1m}	1
2	c_{21}	c_{22}	c_{23}	...	c_{2m}	1
3	c_{31}	c_{32}	c_{33}	...	c_{3m}	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮
m	c_{m1}	c_{m2}	c_{m3}	...	c_{mm}	1
b_j	1	1	1	...	1	m

The table represents that only one unit of job is available for one facility. The assignment is to be made in such a way that each job can be associated with one and only one facility. The problem is to determine ous assignment of job to facilities so as to minimize the over all cost.

Mathematical Formulation of the Problem:-

Assuming that c_{ij} is the cost of assigning the i-th job to the j-th facility, we state the assignment problem mathematically as:

Determine $x_{ij} \geq 0$, $i, j = 1, 2, \ldots, m$.

which optimizes the cost $z = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_{ij}$.

subject to $\sum_{j=1}^{m} x_{ij} = 1$, $i = 1, 2, \ldots, m$ ⟶ ①

and $\sum_{i=1}^{m} x_{ij} = 1$, $j = 1, 2, \ldots, m$ ⟶ ②

The requirement $x_{ij} \geq 0$ in the assignment problem has the explicit form

$$x_{ij} = \begin{cases} 1, & \text{if the i-th job be assigned to the j-th facility,} \\ 0, & \text{otherwise} \end{cases}$$

and as such the assignment problem is not a linear programming problem as the variable x_{ij} can assume only 0 and 1.

The constraint ① insures that only one job is assigned to one person and the constraint ② insures that only one person should be assigned with one job and as such they restrict the possible integral values of each variable to either 0 or 1.

Computational Procedure (Hungarian Method):-

Step-1:- Subtract the minimum element of each row in the cost matrix from the other elements of the respective row. The matrix has now at least one zero in every row. Then subtract the minimum element of each column, which does not have a zero, from the other elements of the respective column to get the starting matrix.

Step-2 :→ Draw the least possible number of horizontal and vertical lines to cover all the zeros of the starting table.

Now two cases may arise:

i) The number of lines so drawn may be equal to the order of the cost matrix, in this case an optimal assignment has been reached.

ii) The number of lines so drawn may be less than the order of the cost matrix.

If the number of lines be equal to the order of the cost matrix we pass on to step-3.

Step-3 :→ Starting with the first row of the starting matrix, examine all the rows of this matrix which contain only one zero in it (row operation). Mark this zero with $\square$ as an assignment will be made there. Draw vertical lines along the columns containing these assigned zeros. This eliminates the possibility of making further assignments in those columns. Examine all the rows in this way.

When all the rows have thus been completely examined, apply similar procedure to the columns successively. In this case start from the first column and examine all the uncrossed columns to find columns containing exactly one remaining zero (column operation). Mark these zeros by $\square$ where an assignment will be made and draw horizontal lines through these marked zeros.

Let us consider the assignment problem represented by the adjacent cost matrix in which the elements represent the times in hours required by a machine to perform the corresponding job. The problem is to allocate the jobs to the machines so as to minimize the total time.

Machines

Jobs	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

Table-1

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Table-2

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Table-3

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Table-4

$\boxed{0}$	14	9	3
9	20	$\boxed{0}$	22
23	$\boxed{0}$	3	0
9	12	14	$\boxed{0}$

Therefore the optimal assignment will be

$A \to I$, $B \to III$, $C \to II$, $D \to IV$, as there is no unmarked zero left.

$\therefore$ The minimum cost (time) is, from the original matrix = sum of cost of the cells (1,1), (2,3), (4,4) and (3,2)

$= (8 + 4 + 19 + 10)$ hours $= 41$ hours.

Following step-2, if the number of lines so drawn be less than the order of the matrix, we pass on to step-4. The lines should be so drawn that a minimum number of them will pass through all the zeros of the matrix.

Step-4: Find the smallest element in the starting table among the uncovered elements left after drawing the lines as given in step-2. Subtract this element from all the uncovered elements of the current matrix and add the same element to the elements lying at the intersection of the horizontal and vertical lines. Do not alter the elements through which only one line passes. This gives the modified matrix with more zeros.

Then go to step-2 with this modified matrix. If a complete assignment be not still available then repeat the steps-4 and 2 iteratively and finally apply the step-3.

Step-5: Repeat the two operations of step-3 (row and column operations) successively until one of the two following cases arise:

i) There will be no unmarked zero left.

ii) There lie more than one unmarked zero in one row or column.

In the first case the algorithm stops and we have exactly one marked zero in each row and in each column of the given matrix. The assignment corresponding to these zeros is the optimal assignment.

In the second case mark with □ one of the unmarked zeros arbitrarily and ignore the remaining zero in that row or column. Repeat the process until no unmarked zero is left in the matrix.

As an example we consider the assignment problem for which the cost matrix is given below. Following step-1, we get the second matrix on the right from the first.

	M_1	M_2	M_3	M_4	M_5
J_1	160	130	175	190	200
J_2	135	120	130	160	175
J_3	140	110	155	170	185
J_4	50	50	80	80	110
J_5	55	35	70	80	105

Table - 1

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

Table - 2

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

Table - 3

15	0	20	15	$\boxed{0}$
15	15	$\boxed{0}$	10	0
15	$\boxed{0}$	20	15	5
$\boxed{0}$	15	20	0	5
5	0	10	$\boxed{0}$	0

The least number of lines required is 5 which is equal to the order of the matrix.

The assignments are made in the cells in the following order: (3,2) by row operation and (4,1), (2,3), (5,4) and (1,5) by column operation. There ~~are~~ is no zero left and every row has an assignment.

Thus the optimal solution will be

$J_1 \to M_5$, $J_2 \to M_3$, $J_3 \to M_2$, $J_4 \to M_1$, $J_5 \to M_4$. and the total cost is $=(200+130+110+50+80)$ unit $= 570$ unit.

Problems :→

1. Find the optimal assignment for a problem with the following cost matrix:

	M_1	M_2	M_3	M_4	M_5
J_1	8	4	2	6	1
J_2	0	9	5	5	4
J_3	3	8	9	2	6
J_4	4	3	1	0	3
J_5	9	5	8	9	5

[Ans: $J_1 \to M_5$, $J_2 \to M_1$, $J_3 \to M_4$, $J_4 \to M_3$, $J_5 \to M_2$, Min. cost = 9]

2. The Head of the department has five jobs A, B, C, D, E and five sub-ordinates V, W, X, Y, Z. The numbers of hours each man would take to perform each job is as follow:

	V	W	X	Y	Z
A	3	5	10	15	8
B	4	7	15	18	8
C	8	12	20	20	12
D	5	5	8	10	6
E	10	10	15	25	10

How would the jobs be allocated to minimize the total time?

[Ans: A→X, B→W, C→V, D→Y, E→Z, Min cost = 45]

3. Find the optimal assignments for the assignment problems with the following cost matrices:

i)

	J_1	J_2	J_3
P_1	12	24	15
P_2	23	18	24
P_3	30	14	28

[Ans: $P_1 \to J_1$, $P_2 \to J_3$, $P_3 \to J_2$, Min cost = 50]

ii)

	1	2	3	4
1	1	4	6	3
2	9	7	10	9
3	4	5	11	7
4	8	7	8	5

[Ans: 1→1, 2→3, 3→2, 4→4, Min cost = 21]

iii)

	M_1	M_2	M_3	M_4
J_1	10	24	30	15
J_2	16	22	28	12
J_3	12	20	32	10
J_4	9	26	34	16

[Ans: $J_1 \to M_3$, $J_2 \to M_2$, $J_3 \to M_4$, $J_4 \to M_1$
or $J_1 \to M_2$, $J_2 \to M_4$, $J_3 \to M_2$, $J_4 \to M_1$
or $J_1 \to M_2$, $J_2 \to M_3$, $J_3 \to M_4$, $J_4 \to M_2$
Min. cost = 71]

iv)

	a	b	c		
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

[Ans: 1→a, 2→d, 3→e, 4→c, 5→b, Min. Cost = 60]